

# **Set 5: Constraint Satisfaction Problems**

## **Chapter 6 R&N**

**ICS 271 Fall 2016**  
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# Outline

- **The constraint network model**
  - Variables, domains, constraints, constraint graph, solutions
- **Examples:**
  - graph-coloring, 8-queen, cryptarithmic, crossword puzzles, vision problems, scheduling, design
- **The search space and naive backtracking,**
- **The constraint graph**
- **Consistency enforcing algorithms**
  - arc-consistency, AC-1, AC-3
- **Backtracking strategies**
  - Forward-checking, dynamic variable orderings
- **Special case: solving tree problems**
- **Local search for CSPs**

## Constraint satisfaction problems (CSPs)

Standard search problem:

**state** is a “black box”—any old data structure  
that supports goal test, eval, successor

CSP:

**state** is defined by *variables*  $X_i$  with *values* from *domain*  $D_i$

**goal test** is a set of *constraints* specifying  
allowable combinations of values for subsets of variables

Simple example of a *formal representation language*

Allows useful *general-purpose* algorithms with more power  
than standard search algorithms

# Constraint Satisfaction

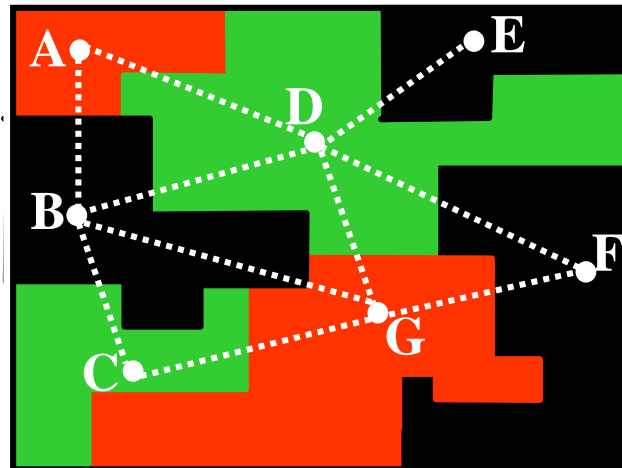
## Example: map coloring

Variables - countries (A,B,C,etc.)

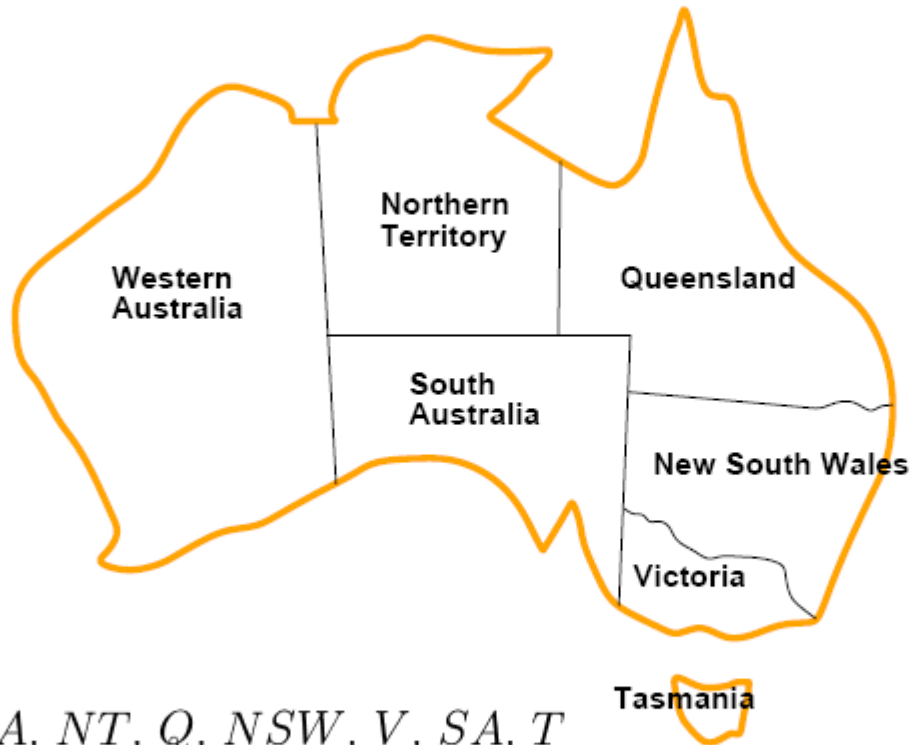
Values - colors (e.g., red, green, yellow)

Constraints:  $A \neq B$ ,  $A \neq D$ ,  $D \neq E$ , etc.

A	B
red	green
red	yellow
green	red
green	yellow
yellow	green
yellow	red



## Example: Map-Coloring



**Variables**  $WA, NT, Q, NSW, V, SA, T$

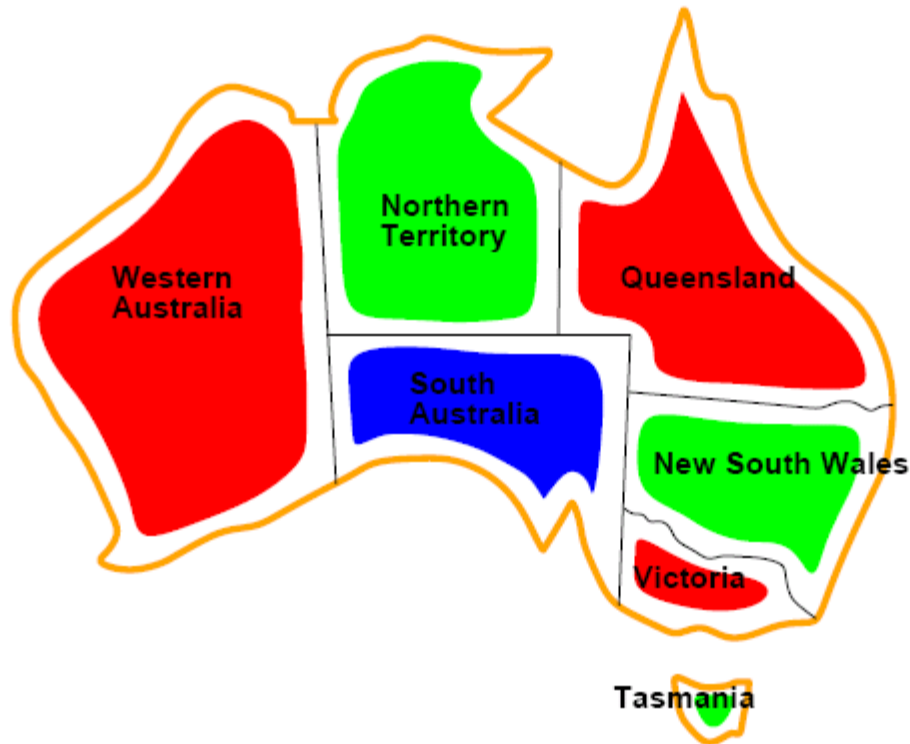
**Domains**  $D_i = \{red, green, blue\}$

**Constraints:** adjacent regions must have different colors

e.g.,  $WA \neq NT$  (if the language allows this), or

$(WA, NT) \in \{(red, green), (red, blue), (green, red), (green, blue), \dots\}$

## Example: Map-Coloring contd.

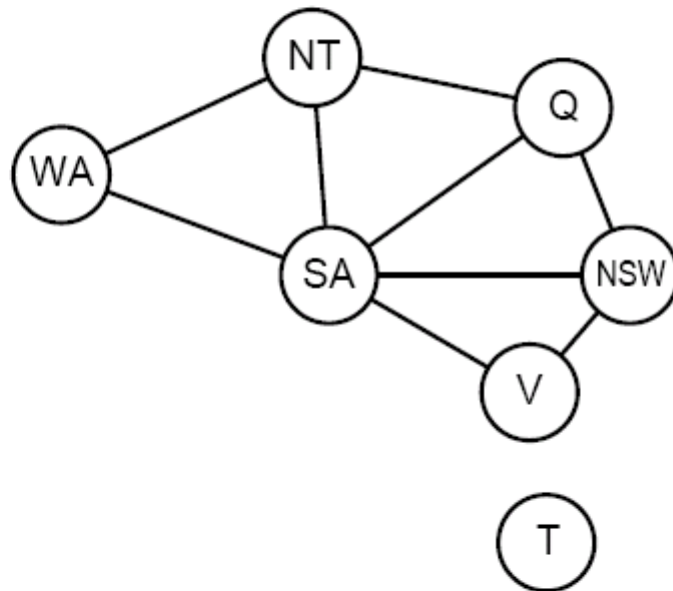


**Solutions** are assignments satisfying all constraints, e.g.,  
 $\{WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green\}$

## Constraint graph

*Binary CSP*: each constraint relates at most two variables

*Constraint graph*: nodes are variables, arcs show constraints



General-purpose CSP algorithms use the graph structure to speed up search. E.g., Tasmania is an independent subproblem!

## Varieties of CSPs

### Discrete variables

finite domains; size  $d \Rightarrow O(d^n)$  complete assignments

- ◇ e.g., Boolean CSPs, incl. Boolean satisfiability (NP-complete)

infinite domains (integers, strings, etc.)

- ◇ e.g., job scheduling, variables are start/end days for each job
- ◇ need a **constraint language**, e.g.,  $StartJob_1 + 5 \leq StartJob_3$
- ◇ **linear** constraints solvable, **nonlinear** undecidable

### Continuous variables

- ◇ e.g., start/end times for Hubble Telescope observations
- ◇ linear constraints solvable in poly time by LP methods



# Varieties of constraints

- **Unary** constraints involve a single variable,
  - e.g.,  $SA \neq \text{green}$
- **Binary** constraints involve pairs of variables,
  - e.g.,  $SA \neq WA$
- **Higher-order** constraints involve 3 or more variables,
  - Clauses in boolean (propositional) logic
  - e.g., cryptarithmic column constraints

# Hard vs Soft Constraints

- Hard constraints : must be satisfied
  - Satisfaction problem
- Soft constraints : capture preferences
  - Optimization problem

# Sudoku

		2	4		6			
8	6	5	1			2		
	1				8	6		9
9				4		8	6	
	4	7				1	9	
	5	8		6				3
4		6	9				7	
		9			4	5	8	1
			3		2	9		

Each row, column and major block must be all different

“Well posed” if it has unique solution: **27 constraints**

# Sudoku

		2	4		6			
8	6	5	1			2		
	1				8	6		9
9				4		8	6	
	4	7				1	9	
	5	8		6				3
4		6	9				7	<del>2</del>
		9			4	5	8	1
			3		2	9		

•Variables: 81 slots

•Domains =  
{1,2,3,4,5,6,7,8,9}

•Constraints:  
•27 not-equal

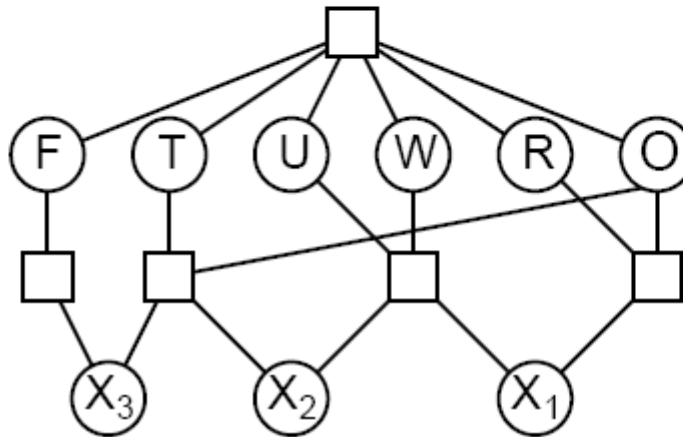
**Constraint  
propagation**

Each row, column and major block must be all different

“Well posed” if it has unique solution: **27 constraints**

## Example: Cryptarithmic

$$\begin{array}{r}
 \text{TWO} \\
 + \text{TWO} \\
 \hline
 \text{FOUR}
 \end{array}$$



**Variables:**  $F T U W R O X_1 X_2 X_3$

**Domains:**  $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

**Constraints**

$alldiff(F, T, U, W, R, O)$

$O + O = R + 10 \cdot X_1$ , etc.

## Real-world CSPs

Assignment problems

e.g., who teaches what class

Timetabling problems

e.g., which class is offered when and where?

Hardware configuration

Spreadsheets

Transportation scheduling

Factory scheduling

Floorplanning

Notice that many real-world problems involve real-valued variables

# A network of constraints

- **Variables**

- $X_1, \dots, X_n$

- **Domains**

- of discrete values:  $D_1, \dots, D_n$

- **Binary constraints:**

- $R_{ij}$  which represent the list of allowed pairs of values,  $R_{ij}$  is a subset of the Cartesian product:  $D_i \times D_j$ .

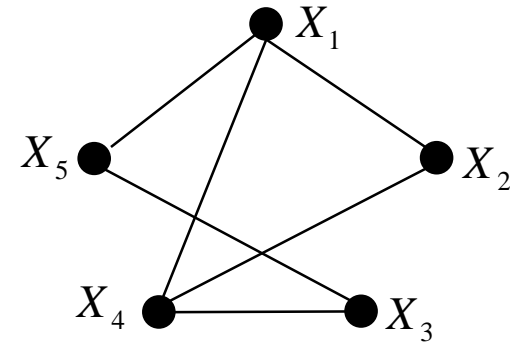
- **Constraint graph:**

- A node for each variable and an edge for each constraint

- **Solution:**

- An assignment of a value from its domain to each variable such that no constraint is violated.

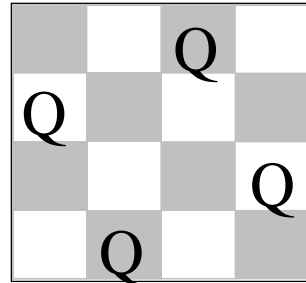
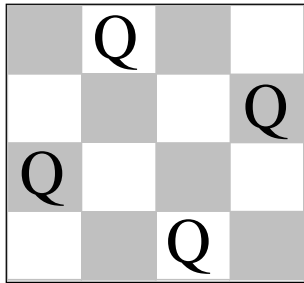
- **A network of constraints represents the relation of all solutions.**



$$sol = \{ (x_1, \dots, x_n) \mid (x_i, x_j) \in R_{ij}, x_i \in D_i, x_j \in D_j \}$$

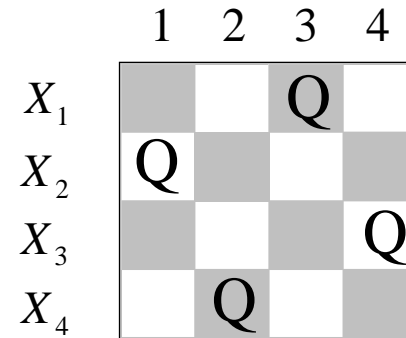
# Example : The 4-queen problem

Place 4 Queens on a chess board of 4x4 such that no two queens reside in the same row, column or diagonal.



Standard CSP formulation of the problem:

- **Variables:** each row is a variable.



- **Domains:**  $D_i = \{1,2,3,4\}$ .

- **Constraints:** There are  $\binom{4}{2} = 6$  constraints involved:

$$R_{12} = \{(1,3)(1,4)(2,4)(3,1)(4,1)(4,2)\}$$

$$R_{13} = \{(1,2)(1,4)(2,1)(2,3)(3,2)(3,4)(4,1)(4,3)\}$$

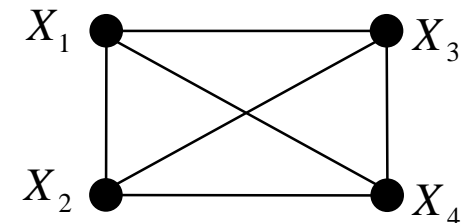
$$R_{14} = \{(1,2)(1,3)(2,1)(2,3)(2,4)(3,1)(3,2)(3,4)(4,2)(4,3)\}$$

$$R_{23} = \{(1,3)(1,4)(2,4)(3,1)(4,1)(4,2)\}$$

$$R_{24} = \{(1,2)(1,4)(2,1)(2,3)(3,2)(3,4)(4,1)(4,3)\}$$

$$R_{34} = \{(1,3)(1,4)(2,4)(3,1)(4,1)(4,2)\}$$

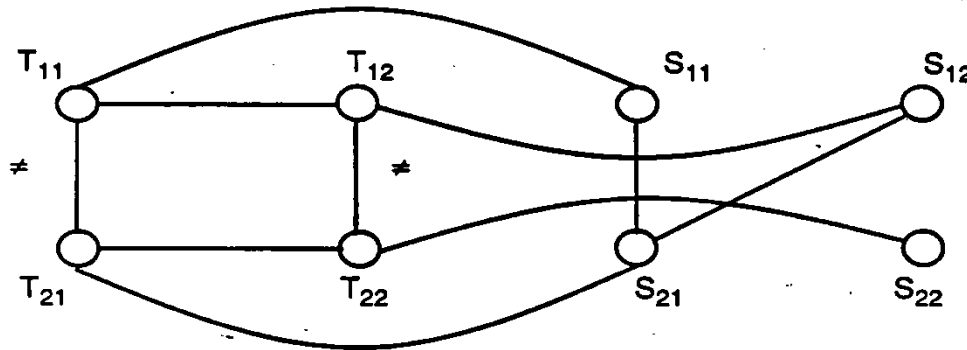
- **Constraint Graph :**





# Class scheduling/Timetabling

- Teachers, Subjects, Classrooms, Time-slots.
- Constraints:
  - A teacher teaches a subset of subjects,
  - Subjects are taught at certain classrooms,
  - A teacher prefers teaching in the morning.
- Task: Assign a teacher and a subject to each class at each time slot, s.t. teachers' happiness is maximized.



$T_{ij}$  - teacher at class  $C_i$  at time  $t_j$        $D(T_{ij}) = \text{teachers}$

$S_{ij}$  - subject taught at class  $C_i$  at time  $t_j$   
 Domain: subjects

# Search vs. Inference

- **Search :**
  - In the space of partial variable-value assignments
  - Key idea : conditioning
- **Inference :**
  - Derive new information implied by values/constraints
  - Key idea : local consistency

## Standard search formulation (incremental)

Let's start with the straightforward, dumb approach, then fix it

States are defined by the values assigned so far

- ◇ **Initial state:** the empty assignment,  $\{ \}$
- ◇ **Successor function:** assign a value to an unassigned variable that does not conflict with current assignment.  
⇒ fail if no legal assignments (not fixable!)
- ◇ **Goal test:** the current assignment is complete

- 1) This is the same for all CSPs!
- 2) Every solution appears at depth  $n$  with  $n$  variables  
⇒ use depth-first search
- 3) Path is irrelevant, so can also use complete-state formulation
- 4)  $b = (n - \ell)d$  at depth  $\ell$ , hence  $n!d^n$  leaves!!!!

But only  $d^n$  unique assignments

## Backtracking search

Variable assignments are **commutative**, i.e.,

$[WA = red \text{ then } NT = green]$  same as  $[NT = green \text{ then } WA = red]$

Only need to consider assignments to a single variable at each node

$\Rightarrow b = d$  and there are  $d^n$  leaves

Depth-first search for CSPs with single-variable assignments  
is called **backtracking** search

Backtracking search is the basic uninformed algorithm for CSPs

Can solve  $n$ -queens for  $n \approx 25$

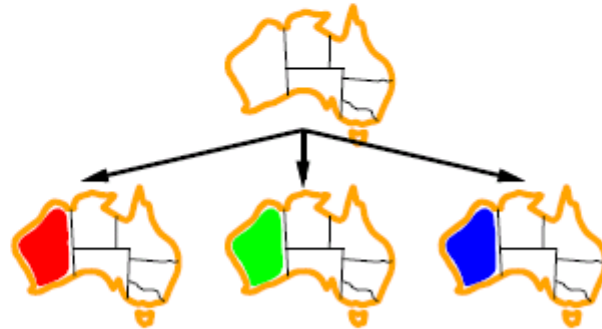
# The search space

- **Definition:** given an ordering of the variables  $X_1, \dots, X_n$ 
  - **a state:**
    - is an assignment to a subset of variables that is consistent.
  - **Operators:**
    - add an assignment to the next variable that does not violate any constraint. *How to choose next variable?*
  - **Goal state:**
    - a consistent assignment to **all** the variables.

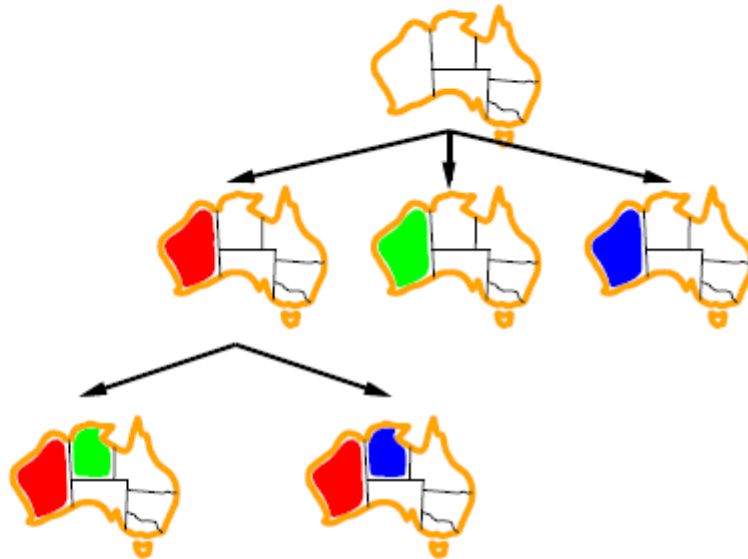
## Backtracking example



## Backtracking example

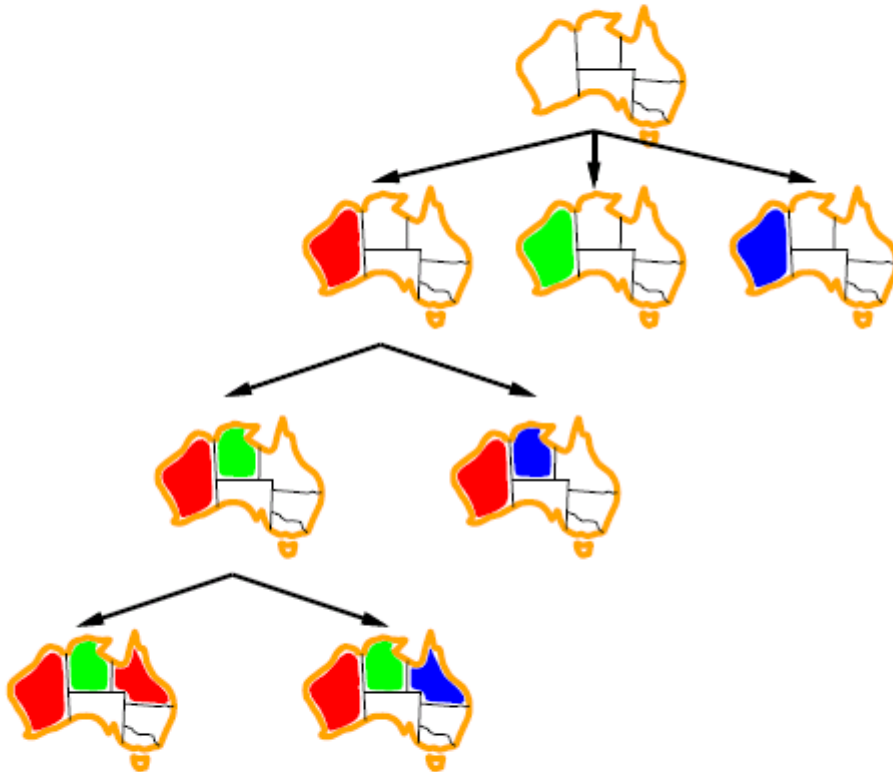


## Backtracking example





## Backtracking example



# Backtracking

```
procedure BACKTRACKING
Input: A constraint network  $P = (X, D, C)$ .
Output: Either a solution, or notification that the network is inconsistent.

   $i \leftarrow 1$                 (initialize variable counter)
   $D'_i \leftarrow D_i$         (copy domain)
  while  $1 \leq i \leq n$ 
    instantiate  $x_i \leftarrow \text{SELECTVALUE}$ 
    if  $x_i$  is null            (no value was returned)
       $i \leftarrow i - 1$     (backtrack)
    else
       $i \leftarrow i + 1$     (step forward)
       $D'_i \leftarrow D_i$ 
    end while
  if  $i = 0$ 
    return "inconsistent"
  else
    return instantiated values of  $\{x_1, \dots, x_n\}$ 
  end procedure

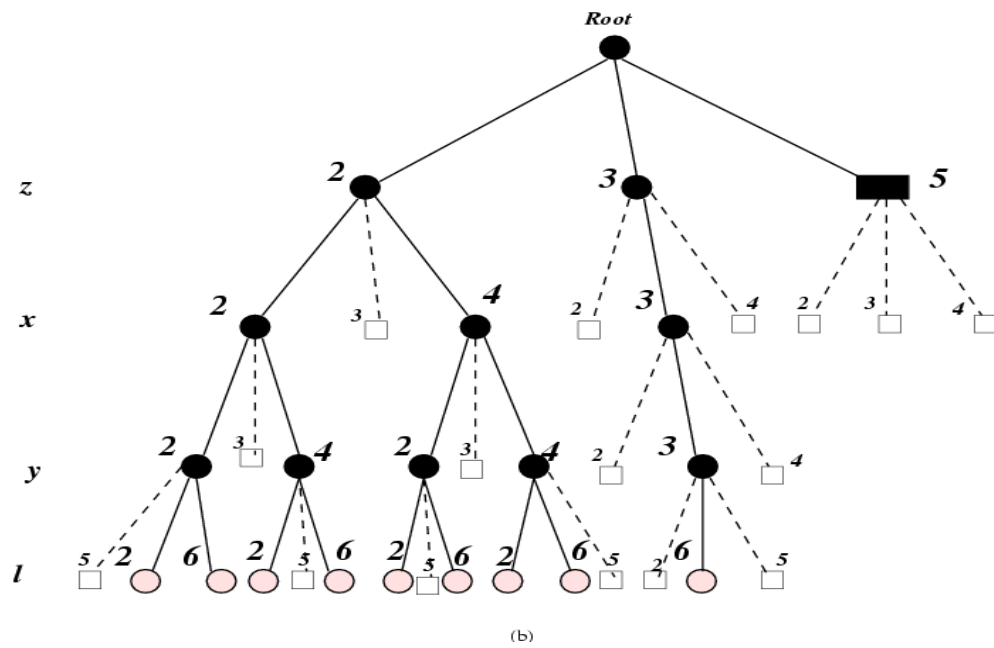
subprocedure SELECTVALUE (return a value in  $D'_i$  consistent with  $\vec{a}_{i-1}$ )

  while  $D'_i$  is not empty
    select an arbitrary element  $a \in D'_i$ , and remove  $a$  from  $D'_i$ 
    if CONSISTENT( $\vec{a}_{i-1}, x_i = a$ )
      return  $a$ 
    end while
  return null                (no consistent value)
end procedure
```

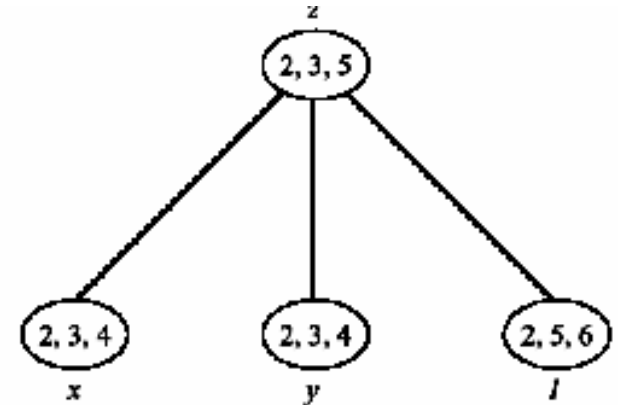
Figure 5.4: The backtracking algorithm.

- **Complexity of extending a partial solution:**
  - **Complexity of consistent:  $O(e \log t)$ ,  $t$  bounds #tuples,  $e$  bounds #constraints**
  - **Complexity of selectvalue:  $O(e k \log t)$ ,  $k$  bounds domain size**

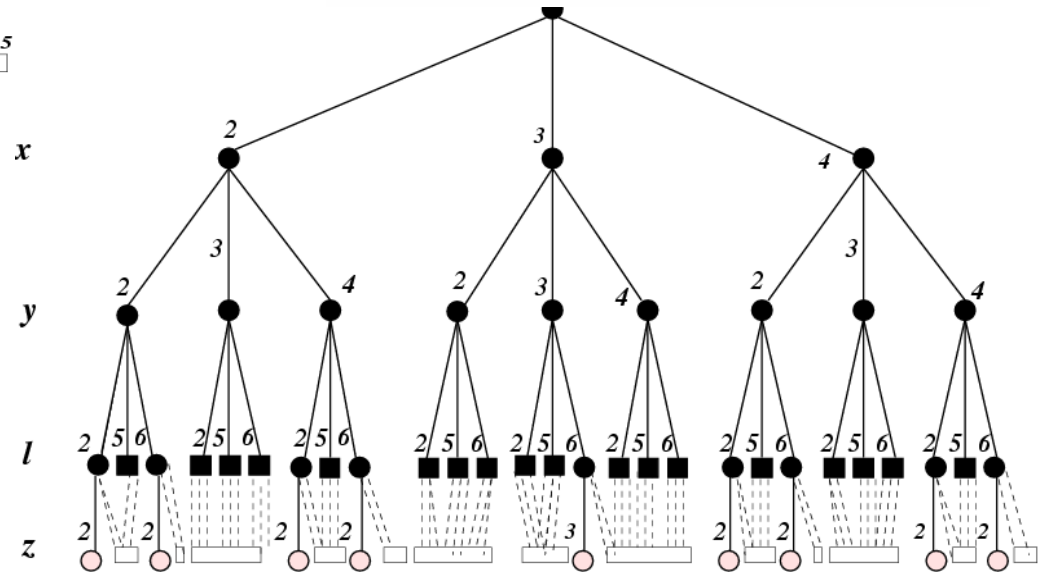
# The effect of variable ordering



$z$  divides  $x, y$  and  $t$



(a)



(c)

## A coloring problem

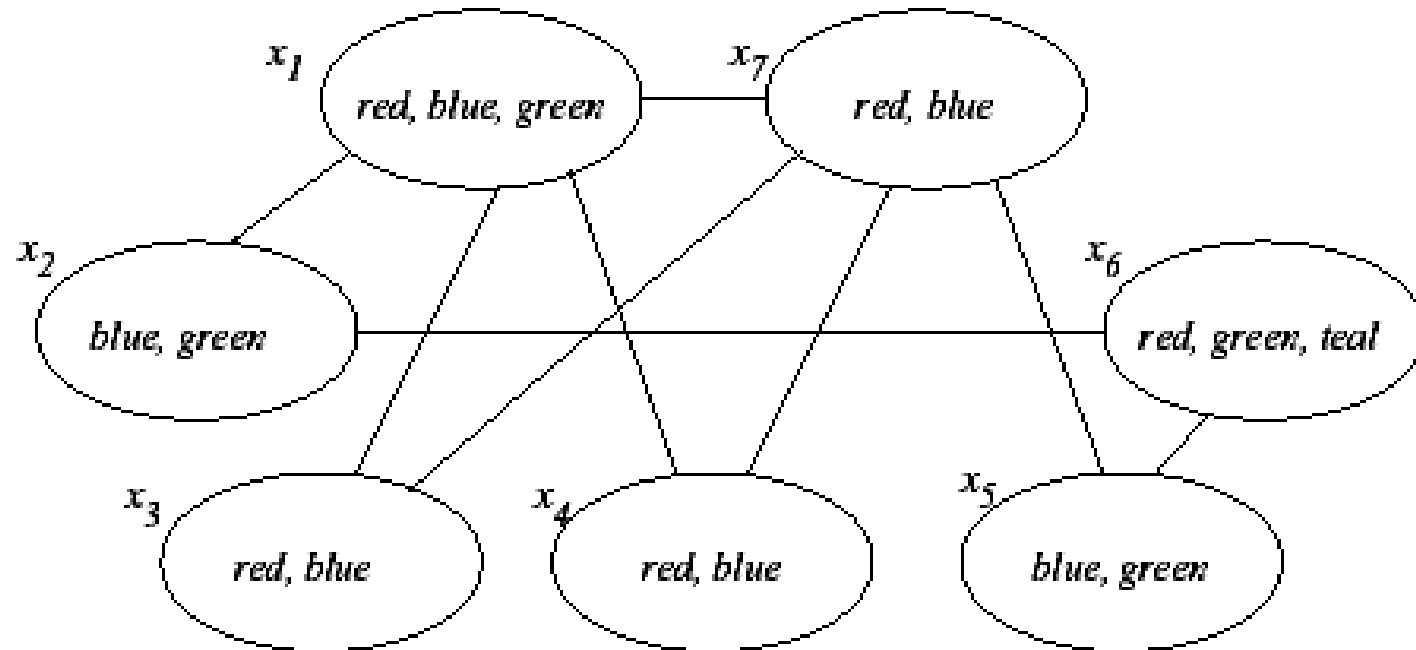
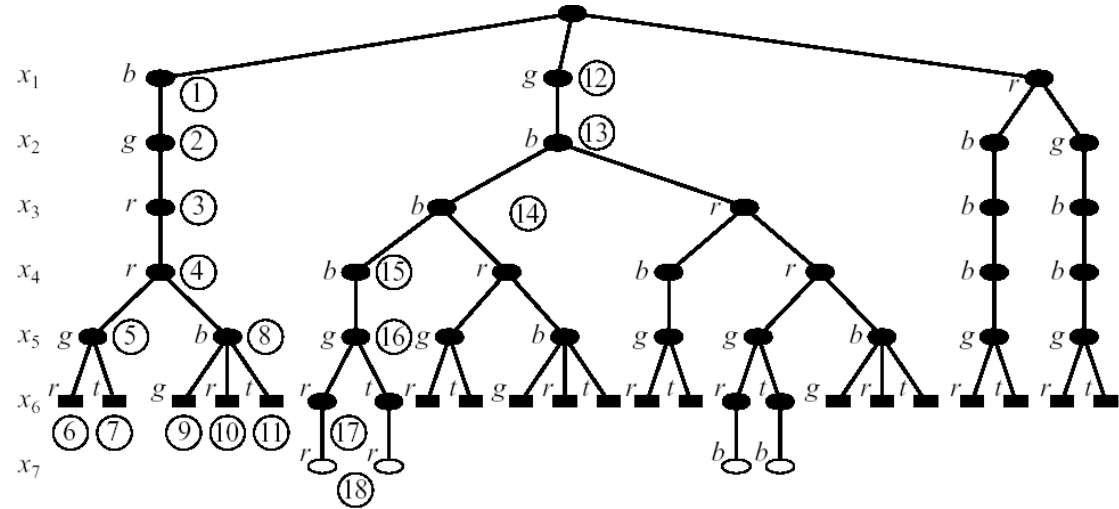
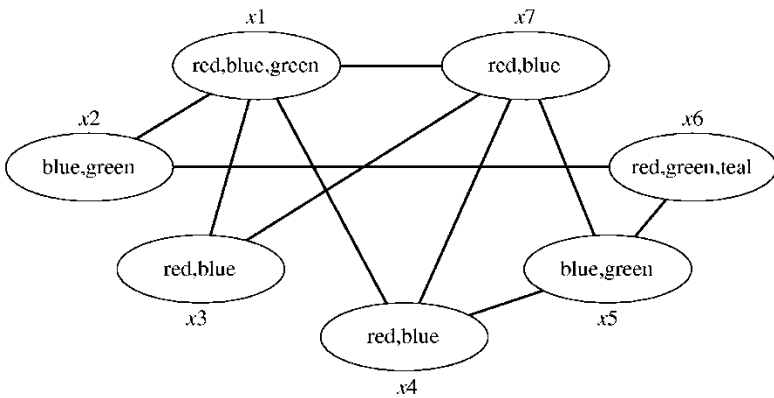
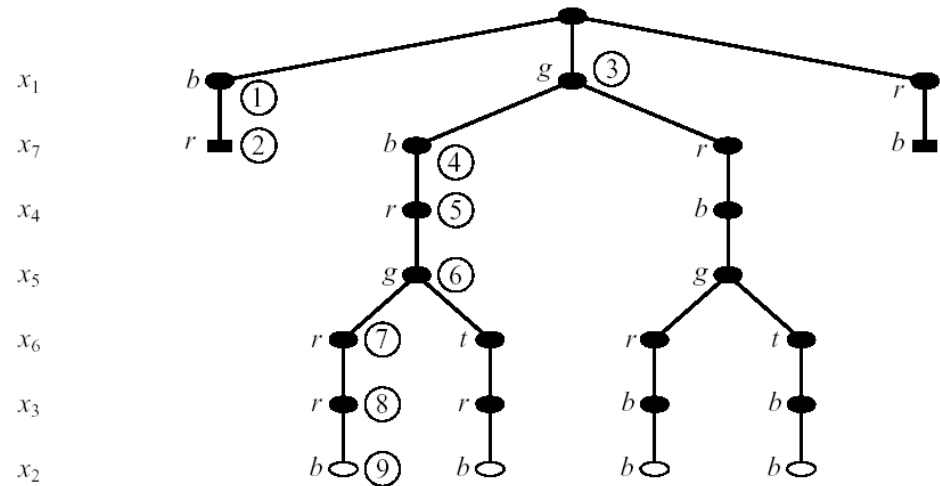


Figure 5.3: A coloring problem with variables  $(x_1, x_2, \dots, x_7)$ . The domain of each variable is written inside the corresponding node. Each arc represents the constraint that the two variables it connects must be assigned different colors.

# Backtracking Search for a Solution



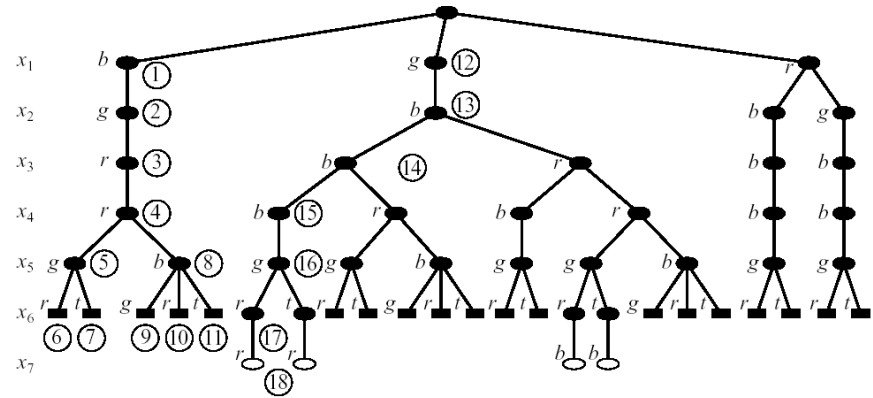
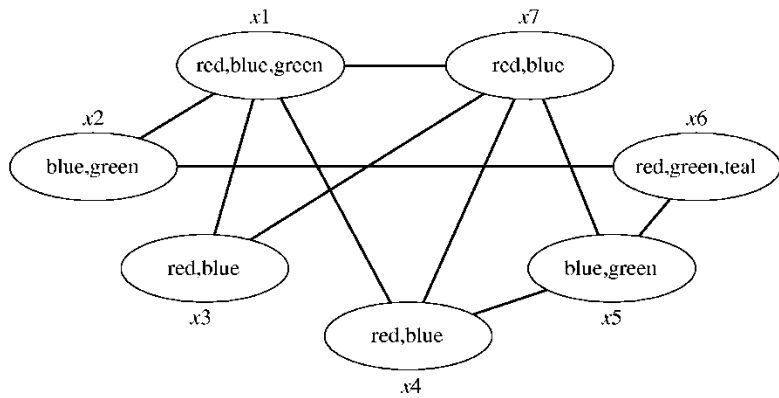
(a)



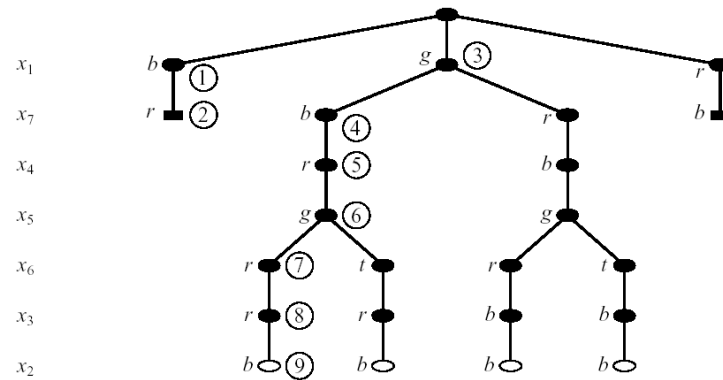
(b)



# Backtracking Search for **All** Solutions



(a)



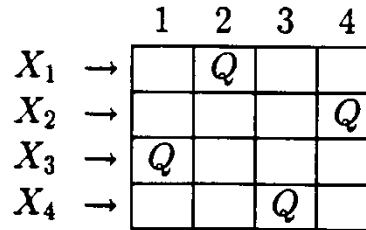
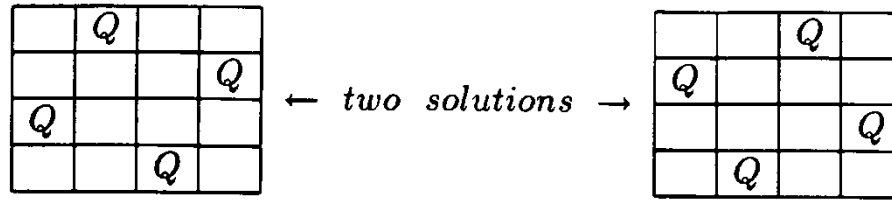
(b)

# Summary so far

- **Constraint Satisfaction Problems**
  - variables, domains, constraints
  - solution = assignment to all variables such that all constraints are satisfied
  - constraint graph
- **Search vs Inference:**
  - conditioning vs local consistency.
- **Search**
  - Backtracking search :
    - In the space of partial assignments;
    - add consistent var=value assignment at each search space node.
    - If not possible, backtrack
  - Effect on search space size on var/val order :
    - Variable order – most constrained variable first
    - Value order – least constraining value first



# The Minimal network: Example: the 4-queen problem



$$\begin{aligned}
 R_{12} &= \{ (1,3) (1,4) (2,4) (3,1) (4,1) (4,2) \} \\
 R_{13} &= \{ (1,2) (1,4) (2,1) (2,3) (3,2) (3,4) (4,1) (4,3) \} \\
 R_{14} &= \{ (1,2) (1,3) (2,1) (2,3) (2,4) (3,1) (3,2) (3,4) (4,2) (4,3) \} \\
 R_{23} &= \{ (1,3) (1,4) (2,4) (3,1) (4,1) (4,2) \} \\
 R_{24} &= \{ (1,2) (1,4) (2,1) (2,3) (3,2) (3,4) (4,1) (4,3) \} \\
 R_{34} &= \{ (1,3) (1,4) (2,4) (3,1) (4,1) (4,2) \}
 \end{aligned}$$

$$\text{the solution } \rho = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 \\ \hline 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \end{pmatrix}$$

$$M_\rho = \text{proj}(\rho) = \left\{ \begin{array}{l} M_{12} = \{ (2,4) (3,1) \} \\ M_{13} = \{ (2,1) (3,4) \} \\ M_{14} = \{ (2,3) (3,2) \} \\ M_{23} = \{ (1,4) (4,1) \} \\ M_{24} = \{ (1,2) (4,3) \} \\ M_{34} = \{ (1,3) (4,2) \} \end{array} \right.$$

# Inference (Approximation) algorithms

- **Arc-consistency (Waltz, 1972)**
- **Path-consistency (Montanari 1974, Mackworth 1977)**
- **k-consistency (Freuder 1982)**
- **Key Idea :**
  - Transform the network into smaller and smaller (but equivalent) networks by tightening the domains/constraints.

# Arc-consistency

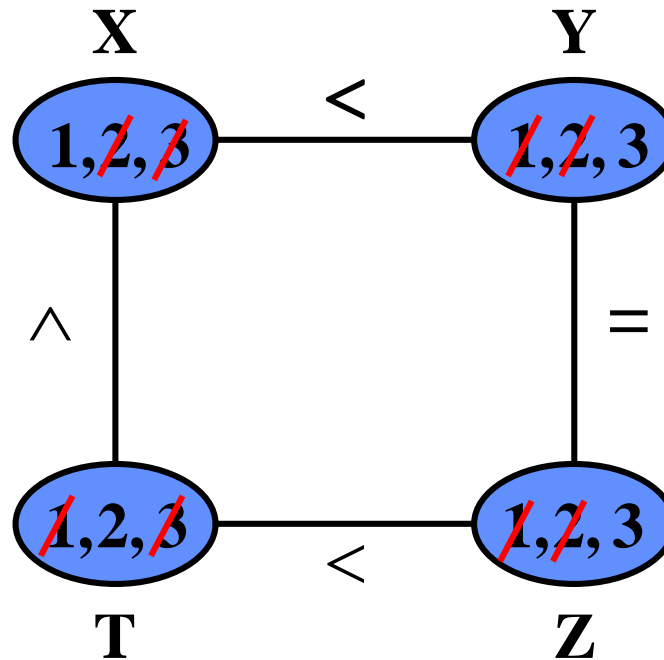
$1 \leq X, Y, Z, T \leq 3$

$X < Y$

$Y = Z$

$T < Z$

$X < T$



# Arc-consistency

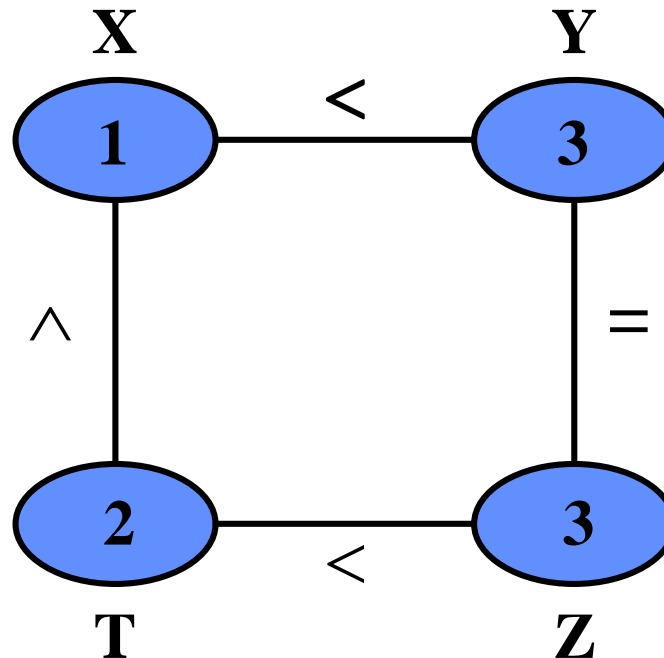
$$1 \leq X, Y, Z, T \leq 3$$

$$X < Y$$

$$Y = Z$$

$$T < Z$$

$$X < T$$

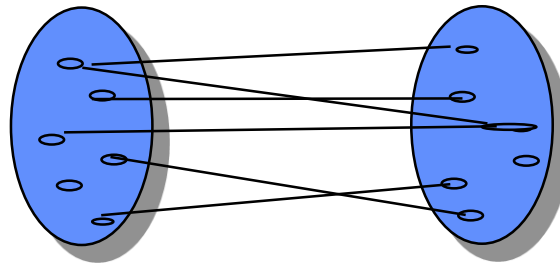


- Incorporated into backtracking search
- Constraint programming languages powerful approach for modeling and solving combinatorial optimization problems.

# Arc-consistency algorithm

domain of X

domain of Y



Arc  $(X_i, X_j)$  is arc-consistent if for any value of  $X_i$ , there exist a matching value of  $X_j$ .

**Algorithm Revise**  $(X_i, X_j)$  makes an arc consistent

Begin

1. For each **a** in  $D_i$  if there is no value **b** in  $D_j$  that is consistent with **a** then delete **a** from the  $D_i$ .

End.

**Revise is  $O(k^2)$ ,  $k$  is the number of values in each domain.**

# Algorithm AC-3

- **Begin**

- 1. Q  $\leftarrow$  put all arcs in the queue in both directions
- 2. While Q is not empty do,
- 3. Select and delete an arc  $(X_i, X_j)$  from the queue Q
  - 4. Revise  $(X_i, X_j)$
  - 5. If Revise changes (reduces) the domain of  $X_i$  then add to the queue all arcs that go to  $X_i$  ( $(X_l, X_i)$  for all  $l$  except  $i$ ).
- 6. end-while

- **End**

- **Complexity:**

- Processing an arc requires  $O(k^2)$  steps
- There is  $e$  edges
- The number of times each arc can be processed is  $2 \cdot k$
- Total complexity is  $O(ek^3)$

# Sudoku

		2	4		6			
8	6	5	1			2		
	1				8	6		9
9				4		8	6	
	4	7				1	9	
	5	8		6				3
4		6	9				7	<del>2</del>
		9			4	5	8	1
			3		2	9		

•Variables: 81 slots

•Domains =  
{1,2,3,4,5,6,7,8,9}

•Constraints:  
•27 not-equal

**Constraint  
propagation**

Each row, column and major block must be all different

“Well posed” if it has unique solution: **27 constraints**

# Sudoku

		2	1	5				6
			3	6	8			1
6	1	8			2			4
		5		2				3
	9	3				5	4	
1				3		6		
3			8			4		7
	8		6	4	3			
5				1	7	9		

Path-consistency or  
3-consistency

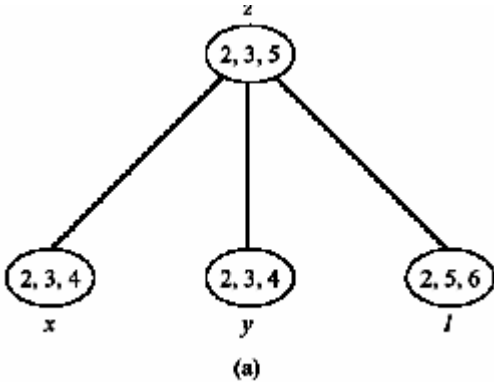
4-consistency and  
i-consistency in general

**Each row, column and major block must be all different**

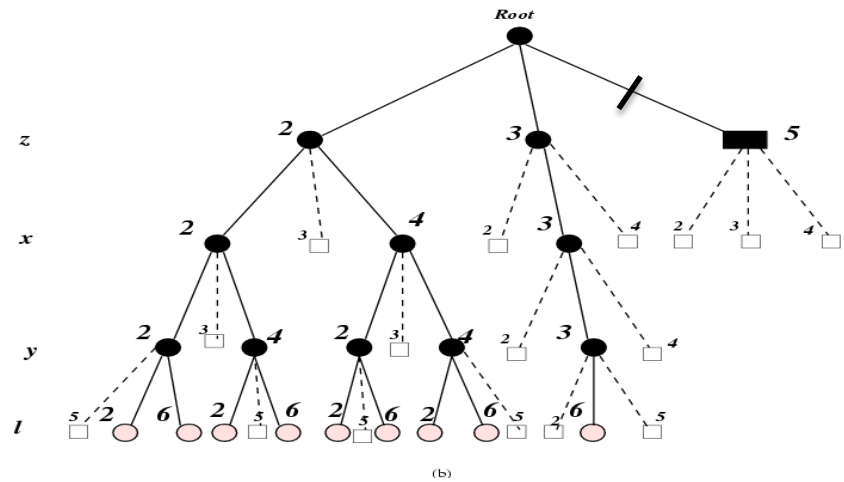
**“Well posed” if it has unique solution**



# The Effect of Consistency Level

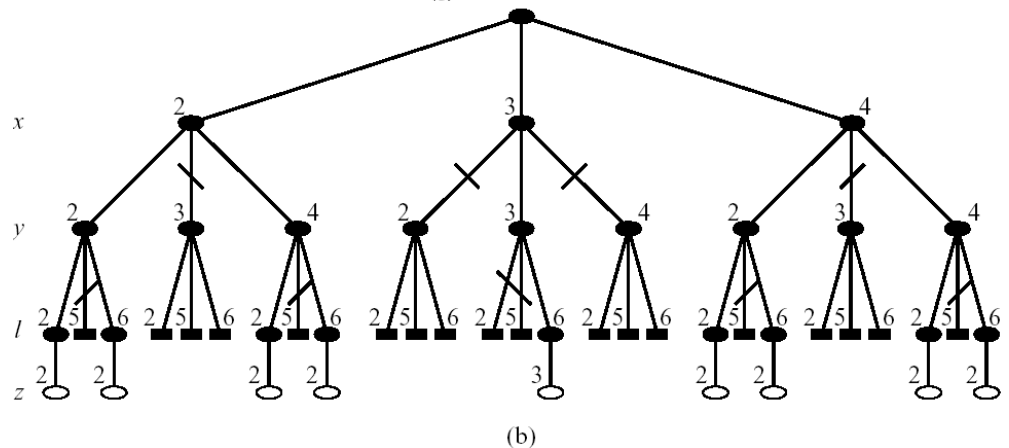


- After arc(2)-consistency  $z=5$  and  $l=5$  are removed



- After path(3)-consistency

- $R'_{zx}$
- $R'_{zy}$
- $R'_{zl}$
- $R'_{xy}$
- $R'_{xl}$
- $R'_{yl}$



**Tighter networks yield smaller search spaces**

# Improving Backtracking $O(\exp(n))$

- **Before search: (reducing the search space)**
  - Arc-consistency, path-consistency, k-consistency
  - Variable ordering (fixed)
- **During search:**
  - **Look-ahead schemes:**
    - **Variable ordering** (*Choose the most constraining variable*)
    - **Value ordering** (*Choose the least restricting value*)
  - **Look-back schemes:**
    - Backjumping
    - Constraint recording
    - Dependency-directed backtracking

## Improving backtracking efficiency

*General-purpose* methods can give huge gains in speed:

1. Which variable should be assigned next?
2. In what order should its values be tried?
3. Can we detect inevitable failure early?
4. Can we take advantage of problem structure?

# Look-ahead: Variable and value orderings

- **Intuition:**
  - Choose a **variable** that will detect failures early
  - Choose a **value** least likely to yield a dead-end
  - Approach: apply propagation at each node in the search tree
- **Forward-checking**
  - Check each unassigned variable separately
- **Maintaining arc-consistency (MAC)**
  - Apply full arc-consistency

## Most constrained variable

Most constrained variable:

choose the variable with the fewest legal values



## Most constraining variable

Tie-breaker among most constrained variables

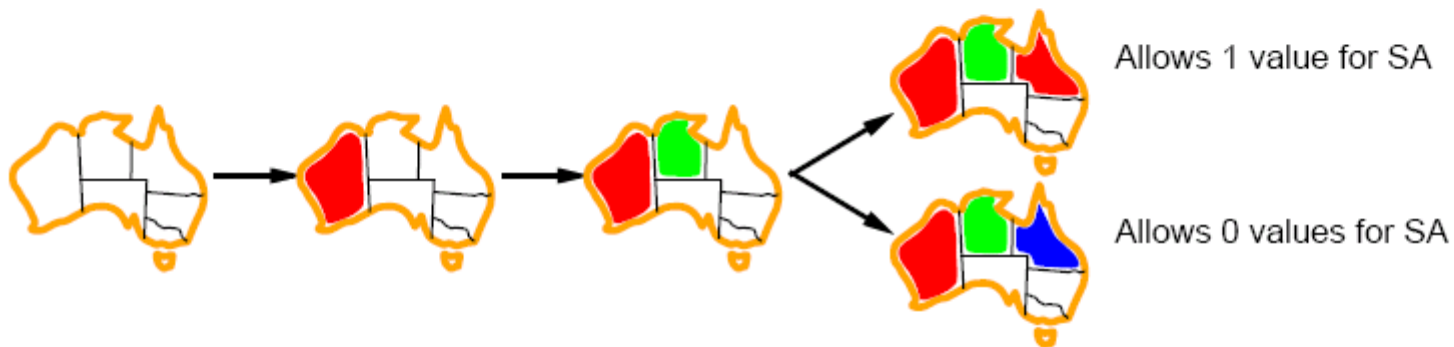
Most constraining variable:

choose the variable with the most constraints on remaining variables



## Least constraining value

Given a variable, choose the least constraining value:  
the one that rules out the fewest values in the remaining variables



Combining these heuristics makes 1000 queens feasible

## Forward checking

Idea: Keep track of remaining legal values for unassigned variables  
Terminate search when any variable has no legal values



WA

NT

Q

NSW

V

SA

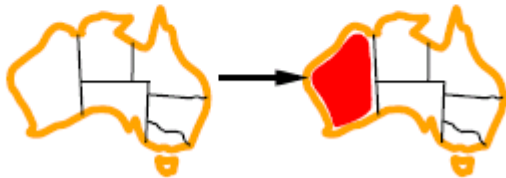
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## Forward checking

Idea: Keep track of remaining legal values for unassigned variables  
Terminate search when any variable has no legal values



# Forward checking

Idea: Keep track of remaining legal values for unassigned variables  
 Terminate search when any variable has no legal values



WA	NT	Q	NSW	V	SA	T
Red, Green, Blue	Red, Green, Blue	Red, Green, Blue	Red, Green, Blue	Red, Green, Blue	Red, Green, Blue	Red, Green, Blue
Red	Green, Blue	Red, Green, Blue	Red, Green, Blue	Red, Green, Blue	Green, Blue	Red, Green, Blue
Red	Blue	Green	Red, Blue	Red, Green, Blue	Blue	Red, Green, Blue

# Forward checking

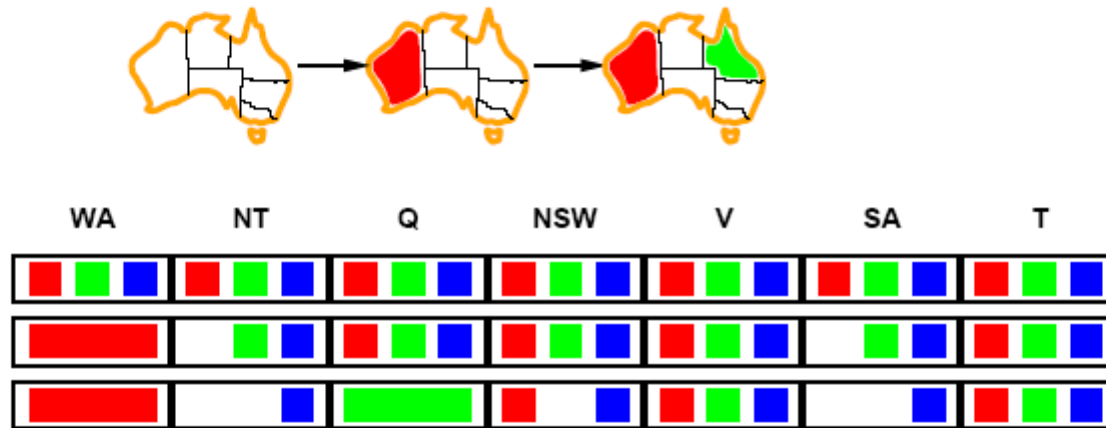
Idea: Keep track of remaining legal values for unassigned variables  
 Terminate search when any variable has no legal values



WA	NT	Q	NSW	V	SA	T
Red, Green, Blue	Red, Green, Blue	Red, Green, Blue	Red, Green, Blue	Red, Green, Blue	Red, Green, Blue	Red, Green, Blue
Red	Green, Blue	Red, Green, Blue	Red, Green, Blue	Red, Green, Blue	Green, Blue	Red, Green, Blue
Red	Blue	Green	Red, Blue	Red, Green, Blue	Blue	Red, Green, Blue
Red	Blue	Green	Red	Blue		Red, Green, Blue

## Constraint propagation

Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:



*NT* and *SA* cannot both be blue!

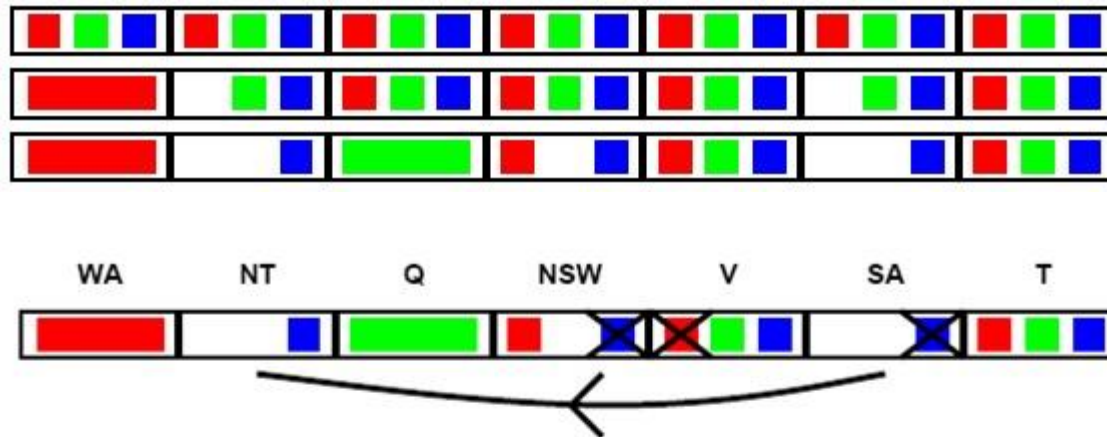
Constraint propagation repeatedly enforces constraints locally

# Arc consistency

Simplest form of propagation makes each arc consistent

$X \rightarrow Y$  is consistent iff

for *every* value  $x$  of  $X$  there is *some* allowed  $y$



If  $X$  loses a value, neighbors of  $X$  need to be rechecked

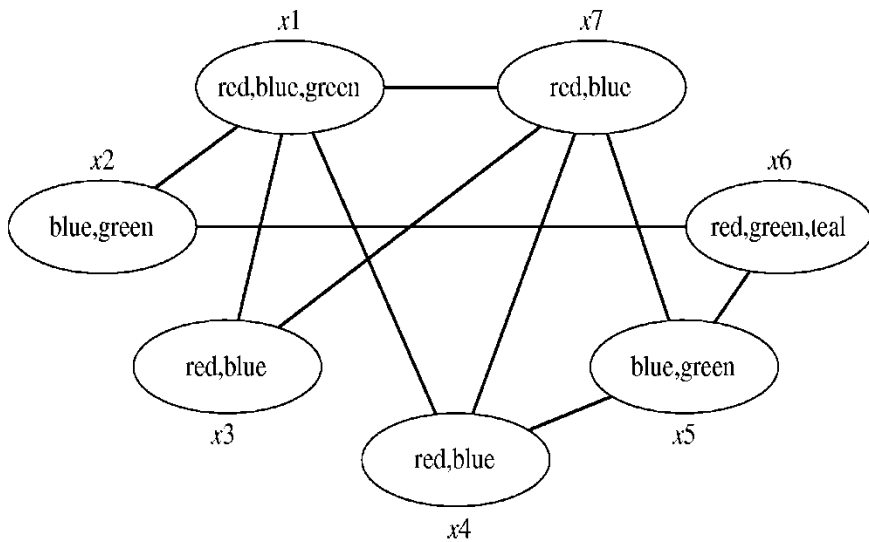
Arc consistency detects failure earlier than forward checking

Can be run as a preprocessor or after each assignment

# Summary so far

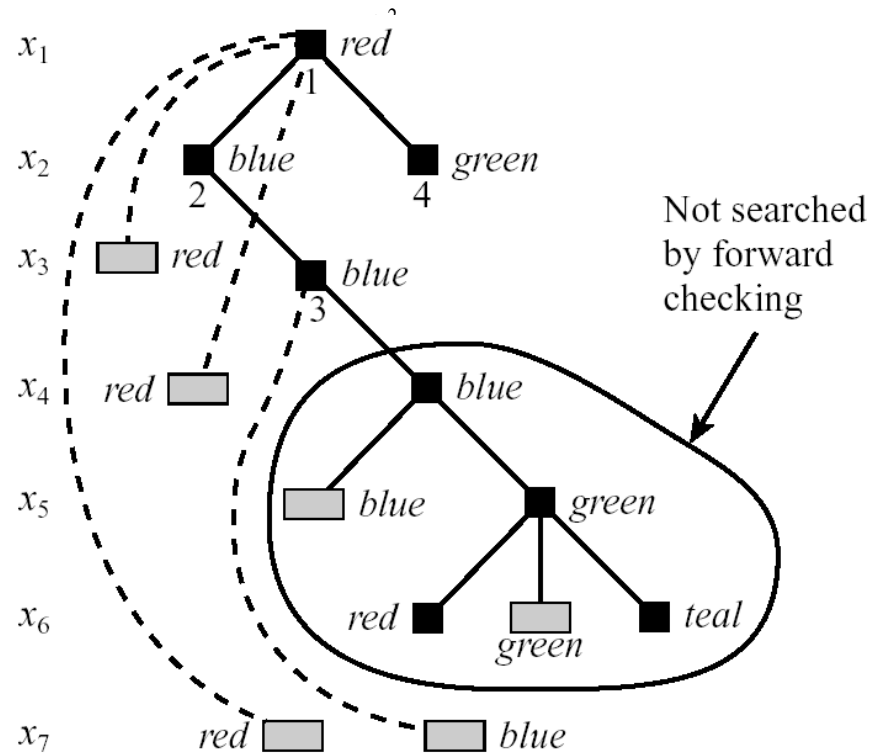
- **Constraint Satisfaction Problems**
  - variables, domains, constraints
  - solution = assignment to all variables such that all constraints are satisfied
  - constraint graph
- **Search vs Inference:**
  - conditioning vs local consistency.
- **Search**
  - In the space of partial assignments; add var=val at each search space node.
  - Effect on search space size on var/val order :
    - Variable order – most constrained variable first
    - Value order – least constraining value first
- **Inference (local consistency)**
  - Prune domains/constraints
  - Arc/path/k-consistency
- **Combining search and local consistency to improve efficiency**
  - Forward checking (FC)
  - Maintaining Arc Consistency (MAC)

# Forward-checking on Graph-coloring



**FW overhead:**  $O(ek^2)$

**MAC overhead:**  $O(ek^3)$



# Algorithm DVO (DVFC)

```
procedure DVFC
Input: A constraint network  $\mathcal{R} = (X, D, C)$ 
Output: Either a solution, or notification that the network is inconsistent.
   $D'_i \leftarrow D_i$  for  $1 \leq i \leq n$     (copy all domains)
   $i \leftarrow 1$                           (initialize variable counter)
   $s = \min_{i < j \leq n} |D'_j|$  (find future var with smallest domain)
   $x_{i+1} \leftarrow x_s$  (rearrange variables so that  $x_s$  follows  $x_i$ )
  while  $1 \leq i \leq n$ 
    instantiate  $x_i \leftarrow \text{SELECTVALUE-FORWARD-CHECKING}$ 
    if  $x_i$  is null (no value was returned)
      reset each  $D'$  set to its value before  $x_i$  was last instantiated
       $i \leftarrow i - 1$  (backtrack)
    else
      if  $i < n$ 
         $i \leftarrow i + 1$  (step forward to  $x_s$ )
         $s = \min_{i < j \leq n} |D'_j|$  (find future var with smallest domain)
         $x_{i+1} \leftarrow x_s$  (rearrange variables so that  $x_s$  follows  $x_i$ )
         $i \leftarrow i + 1$  (step forward to  $x_s$ )
    end while
  if  $i = 0$ 
    return "inconsistent"
  else
    return instantiated values of  $\{x_1, \dots, x_n\}$ 
end procedure
```

Figure 5.12: The DVFC algorithm. It uses the SELECTVALUE-FORWARD-CHECKING sub-procedure given in Fig. 5.8.



# Propositional Satisfiability

## Example: party problem

- If Alex goes, then Becky goes:  $\mathbf{A} \rightarrow \mathbf{B}$  (or,  $\neg\mathbf{A} \vee \mathbf{B}$ )
- If Chris goes, then Alex goes:  $\mathbf{C} \rightarrow \mathbf{A}$  (or,  $\neg\mathbf{C} \vee \mathbf{A}$ )
- Query:

*Is it possible that Chris goes to the party but Becky does not?*



*Is propositional theory*

$\varphi = \{ \neg\mathbf{A} \vee \mathbf{B}, \neg\mathbf{C} \vee \mathbf{A}, \neg\mathbf{B}, \mathbf{C} \}$  satisfiable?

# Unit Propagation

- **Arc-consistency for CNFs.**
- **Involve a single clause and a single literal**
- **Example:**  $(\neg C \vee \mathbf{A}), C \rightarrow \mathbf{A}$
- **Formula:**  $\varphi = \{\neg \mathbf{A} \vee \mathbf{B}, \neg \mathbf{C} \vee \mathbf{A}, \neg \mathbf{B}, \mathbf{C}\}$ 
  - C must be true
  - A must be true
  - B must be false and true
  - formula is not satisfiable

# Look-ahead for SAT

(Davis-Putnam, Logeman and Laveland, 1962)

**DPLL( $\varphi$ )**

**Input:** A cnf theory  $\varphi$

**Output:** A decision of whether  $\varphi$  is satisfiable.

1. Unit\_propagate( $\varphi$ );
2. If the empty clause is generated, return(*false*);
3. Else, if all variables are assigned, return(*true*);
4. Else
5.      $Q$  = some unassigned variable;
6.     return( **DPLL( $\varphi \wedge Q$ )**  $\vee$   
              **DPLL( $\varphi \wedge \neg Q$ )** )

Figure 5.13: The DPLL Procedure



# Look-back: Backjumping / Learning

- **Backjumping:**
  - In deadends, go back to the most recent culprit.
- **Learning:**
  - constraint-recording, no-good recording.
  - good-recording

# Backjumping

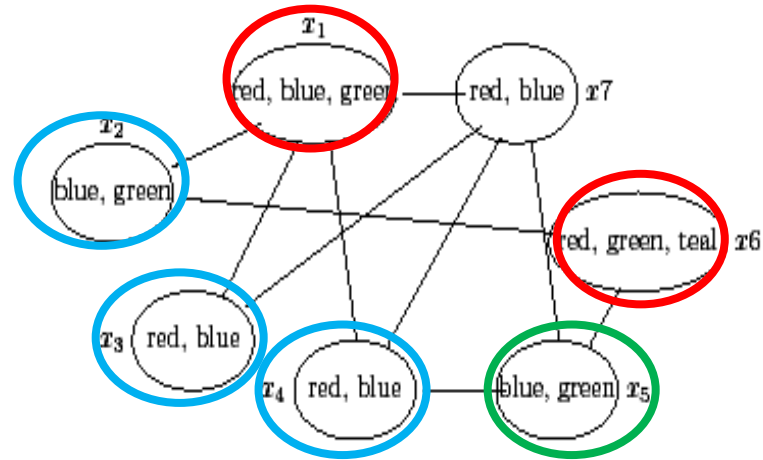
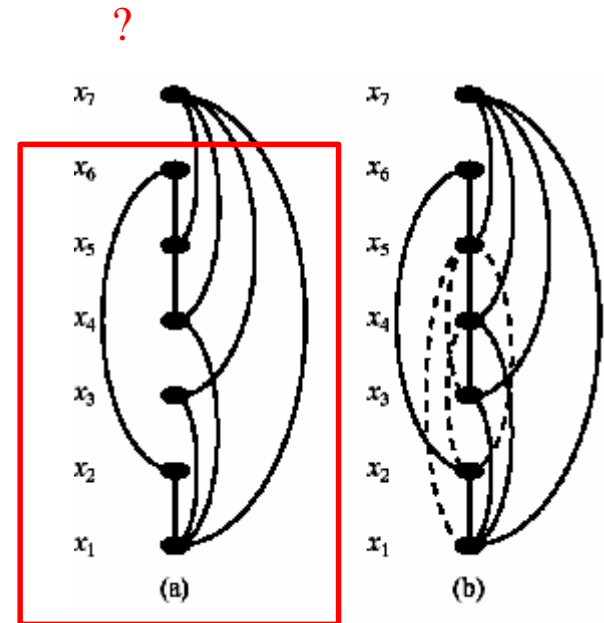
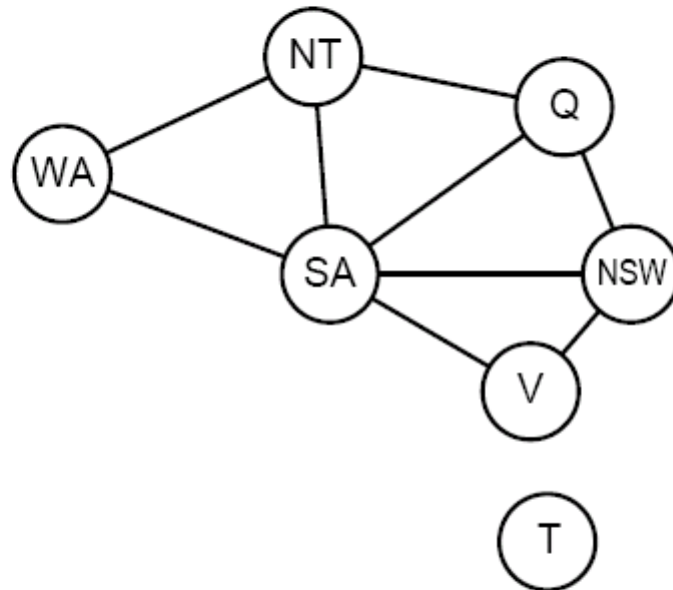


Figure 6.1: A modified coloring problem.

- $(X1=r, x2=b, x3=b, x4=b, x5=g, x6=r, x7=\{r, b\})$
- $(r, b, b, b, g, r)$  **conflict set** of  $x7$ 
  - Conflict set : a partial assignment that cannot be extended to the next variable
- $(r, -, b, b, g, -)$  **c.s. of  $x7$**
- $(r, -, b, -, -, -)$  **minimal conflict-set**
- **Leaf deadend**:  $(r, b, b, b, g, r)$ 
  - No-good : a partial assignment that cannot be extended to a solution
  - No-good = **internal deadend**
- **Every conflict-set is a no-good**
  - **But not other way around ( $x1=r$  is a no-good)**



## Problem structure



Tasmania and mainland are **independent subproblems**

Identifiable as **connected components** of constraint graph

## Problem structure contd.

Suppose each subproblem has  $c$  variables out of  $n$  total

Worst-case solution cost is  $n/c \cdot d^c$ , *linear* in  $n$

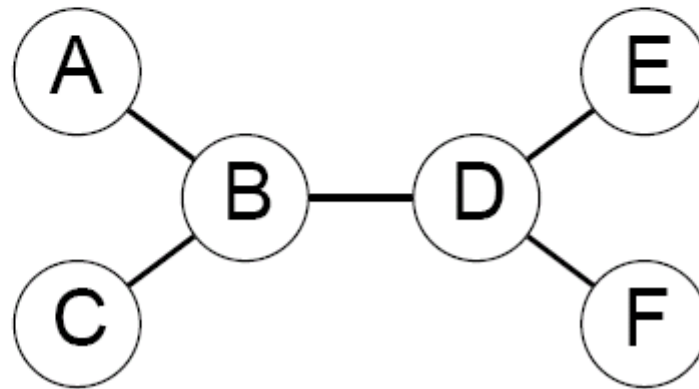
E.g.,  $n = 80$ ,  $d = 2$ ,  $c = 20$

$2^{80} = 4$  billion years at 10 million nodes/sec

$4 \cdot 2^{20} = 0.4$  seconds at 10 million nodes/sec



## Tree-structured CSPs



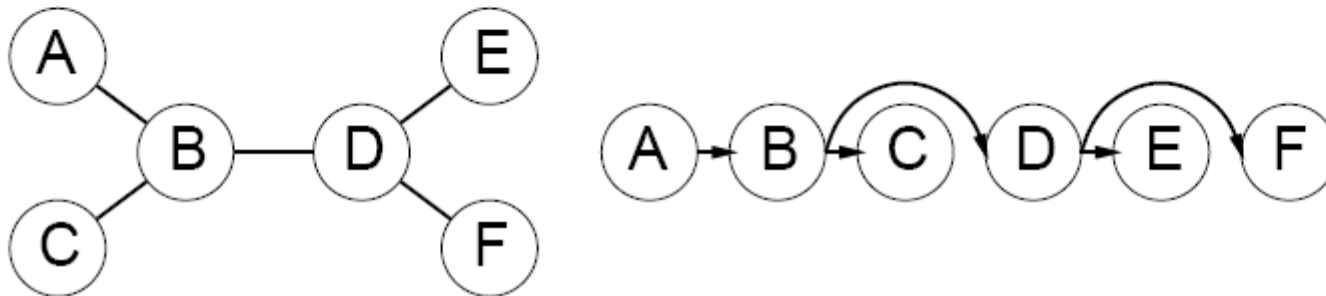
**Theorem:** if the constraint graph has no loops, the CSP can be solved in  $O(n d^2)$  time

Compare to general CSPs, where worst-case time is  $O(d^n)$

This property also applies to logical and probabilistic reasoning:  
an important example of the relation between syntactic restrictions  
and the complexity of reasoning.

## Algorithm for tree-structured CSPs

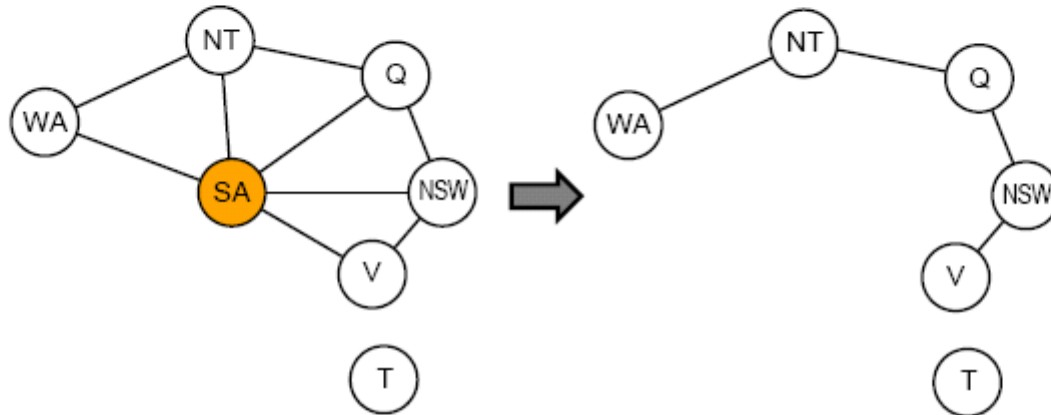
1. Choose a variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering



2. For  $j$  from  $n$  down to 2, apply  $\text{REMOVEINCONSISTENT}(Parent(X_j), X_j)$
3. For  $j$  from 1 to  $n$ , assign  $X_j$  consistently with  $Parent(X_j)$
4. Arc-consistency in tree-structured CSPs makes search backtrack-free

## Nearly tree-structured CSPs

**Conditioning:** instantiate a variable, prune its neighbors' domains



**Cutset conditioning:** instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree

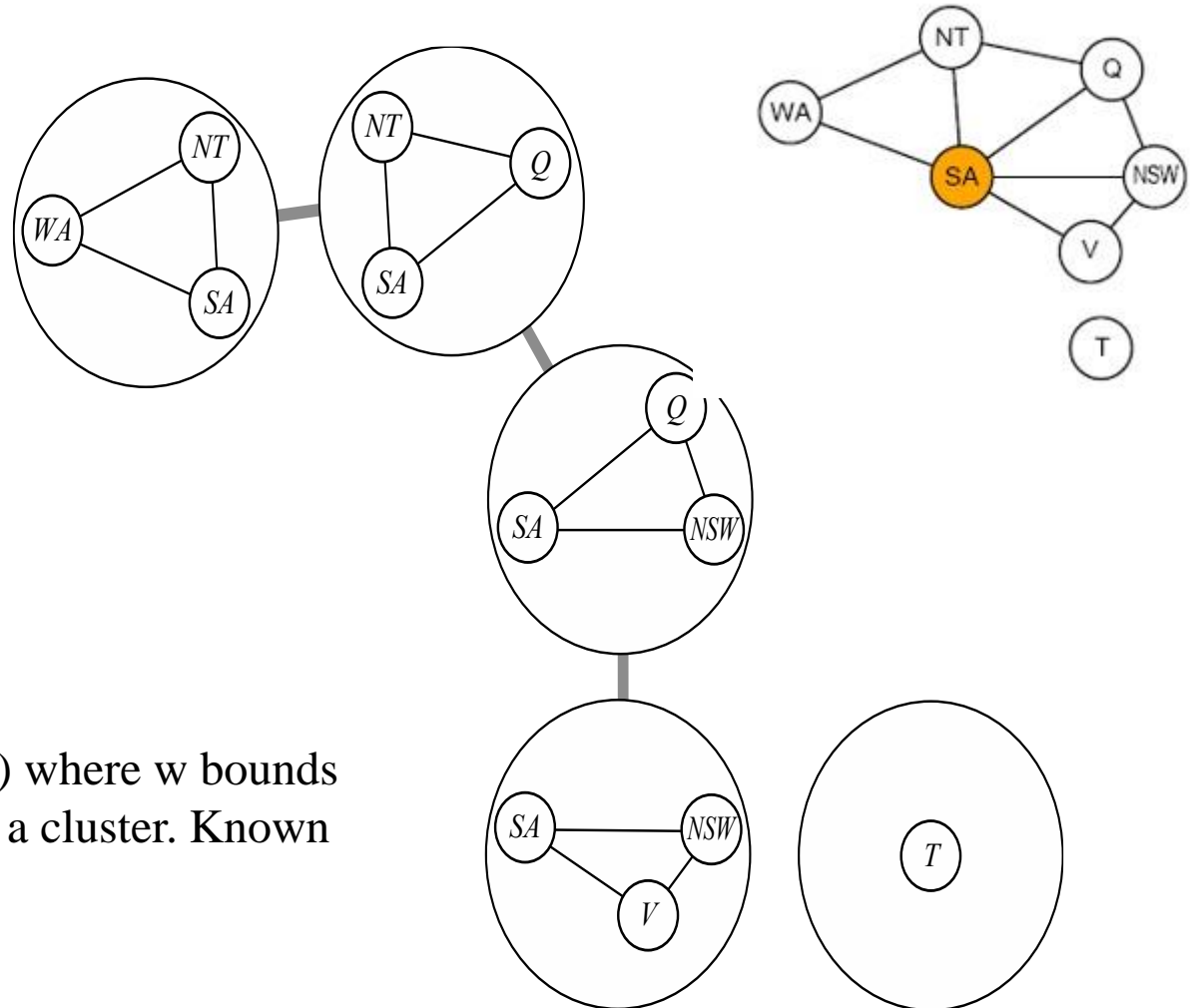
Cutset size  $c \Rightarrow$  runtime  $O(d^c \cdot (n - c)d^2)$ , very fast for small  $c$

# The cycle-cutset method

- An instantiation can be viewed as blocking cycles in the graph
- Given an instantiation to a set of variables that cut all cycles (a cycle-cutset) the rest of the problem can be solved in linear time by a tree algorithm.
- Complexity (n number of variables, k the domain size and C the cycle-cutset size):

$$O(nk^C k^2)$$

# Tree Decomposition



Complexity is  $O(n \exp(w))$  where  $w$  bounds the number of variables in a cluster. Known as the treewidth.

# Local Search for CSPs

- **Local Search**
  - Works with complete variable assignments (called a state)
  - Has a cost function associated with each state
    - E.g. **min-conflicts** – number of constraints violated
  - Considers only immediate neighborhood for next state
    - E.g. change the value of one variable
- **Operators**
  - Greedy heuristic : pick a value for a variable that leads to greatest reduction in total cost – can get stuck in local optima
  - Random moves : occasionally make a random move
  - Sideways moves, re-starts, constraint re-weighting, ...
- **Incomplete**
- **Often fast**
  - orders of magnitude faster than systematic e.g. DFS

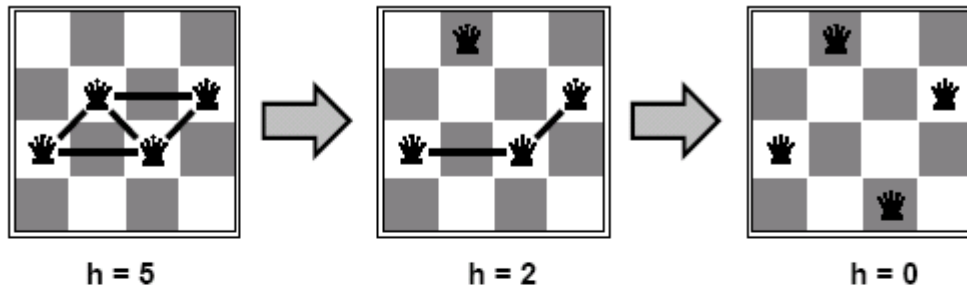
## Example: 4-Queens

**States:** 4 queens in 4 columns ( $4^4 = 256$  states)

**Operators:** move queen in column

**Goal test:** no attacks

**Evaluation:**  $h(n) =$  number of attacks



# GSAT – local search for SAT

(Selman, Levesque and Mitchell, 1992)

1. For i=1 to MaxTries
2.     Select a random assignment A
3.     For j=1 to MaxFlips
4.         if A satisfies all constraints, return A
5.         else flip a variable to maximize the score
6.             (number of satisfied constraints; if no variable
7.             assignment increases the score, flip at random)
8.     end
9. end

Greatly improves hill-climbing by adding  
**restarts** and **sideway moves**



# WalkSAT

(Selman, Kautz and Cohen, 1994)

Adds random walk to GSAT:

With probability  $p$

random walk – flip a variable in some unsatisfied constraint

With probability  $1-p$

perform a hill-climbing step

Randomized hill-climbing often solves  
large and hard satisfiable problems

# More Stochastic Search: Simulated Annealing, Constraint Reweighting

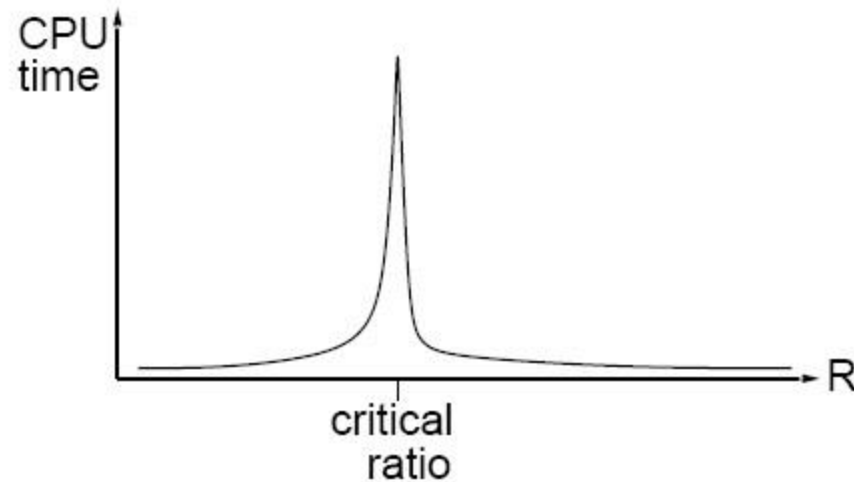
- **Simulated annealing:**
  - A method for overcoming local minimas
  - Allows bad moves with some probability:
    - With some probability related to a temperature parameter  $T$  the next move is picked randomly.
  - Theoretically, with a slow enough cooling schedule, this algorithm will find the optimal solution. But so will searching randomly.
- **Breakout method (Morris, 1990): adjust the weights of the violated constraints**

# Problem Hardness

Given random initial state, can solve  $n$ -queens in almost constant time for arbitrary  $n$  with high probability (e.g.,  $n = 10,000,000$ )

The same appears to be true for any randomly-generated CSP except in a narrow range of the ratio

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$



## Summary

CSPs are a special kind of problem:

- states defined by values of a fixed set of variables
- goal test defined by *constraints* on variable values

Backtracking = depth-first search with one variable assigned per node

Variable ordering and value selection heuristics help significantly

Forward checking prevents assignments that guarantee later failure

Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies

The CSP representation allows analysis of problem structure

Tree-structured CSPs can be solved in linear time

Iterative min-conflicts is usually effective in practice

# Game of Mastermind

## Sample Mastermind game: Game 1

Red	Green	Yellow
-----	-------	--------

Hidden Combination

Guess 2	<table border="1"><tr><td>Red</td><td>Blue</td><td>Green</td></tr></table>	Red	Blue	Green	Correct Position: 1 Correct Colour: 1
Red	Blue	Green			
Guess 1	<table border="1"><tr><td>Red</td><td>Red</td><td>Red</td></tr></table>	Red	Red	Red	Correct Position: 1 Correct Colour: 0
Red	Red	Red			

Variables:  $P_1, \dots, P_n$  one variable for each peg  
Domains: R,G,B,Y

Constraints:  $\{P_1, P_2, P_3: (Red, Green, Yellow), (Red, Yellow, Blue), (Green, Red, Green), (Blue, Red, Green), (Yellow, Red, Green), (Blue, Blue, Red), (Yellow, Blue, Red)\}$